



Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate 2012

Marking Scheme

Mathematics
(Project Maths – Phase 3)

Higher Level

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Introduction

The Higher Level Mathematics examination for candidates in the 24 initial schools for *Project Maths* shared some content with the examination for all other candidates. The marking scheme used for the shared content was identical for the two groups.

This document contains the complete marking scheme for both papers for the candidates in the 24 schools.

Readers should note that, as with all marking schemes used in the state examinations, the detail required in any answer is determined by the context and the manner in which the question is asked, and by the number of marks assigned to the question or part. Requirements and mark allocations may vary from year to year.



Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination, 2012

Mathematics (Project Maths – Phase 3)

Paper 1

Higher Level

Friday 8 June Afternoon 2:00 – 4:30

300 marks

Model Solutions – Paper 1

Note: the model solutions for each question are not intended to be exhaustive – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

Instructions

There are **two** sections in this examination paper:

| | | | |
|-----------|---------------------------|-----------|-------------|
| Section A | Concepts and Skills | 150 marks | 6 questions |
| Section B | Contexts and Applications | 150 marks | 3 questions |

Answer all nine questions.

Write your answers in the spaces provided in this booklet. You will lose marks if you do not do so. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Answer **all six** questions from this section.

Question 1

(25 marks)

(a) Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$

$$a + 2b + 1 = 0$$

$$a = -2b - 1$$

$$(-2b - 1)^2 + (2b + 1)b + b^2 = 3$$

$$7b^2 + 5b - 2 = 0$$

$$(7b - 2)(b + 1) = 0$$

$$b = \frac{2}{7} \quad \text{or} \quad b = -1$$

$$a = \frac{-11}{7} \quad \text{or} \quad a = 1$$

Solution: $\{b = \frac{2}{7} \text{ and } a = \frac{-11}{7}\}$ or $\{b = -1 \text{ and } a = 1\}$.

(b) Find the set of all real values of x for which $\frac{2x-5}{x-3} \leq \frac{5}{2}$.

Multiply across by $2(x-3)^2$, which is non-negative:

$$2(x-3)(2x-5) \leq 5(x-3)^2$$

$$4x^2 - 22x + 30 \leq 5x^2 - 30x + 45$$

$$0 \leq x^2 - 8x + 15$$

$$0 \leq (x-5)(x-3)$$

$$x \geq 5 \text{ or } x < 3.$$

OR

$$\frac{2x-5}{x-3} - \frac{5}{2} \leq 0$$

$$\frac{2(2x-5) - 5(x-3)}{2(x-3)} \leq 0$$

$$\frac{-x+5}{2(x-3)} \leq 0$$

$$x \geq 5 \text{ or } x < 3.$$

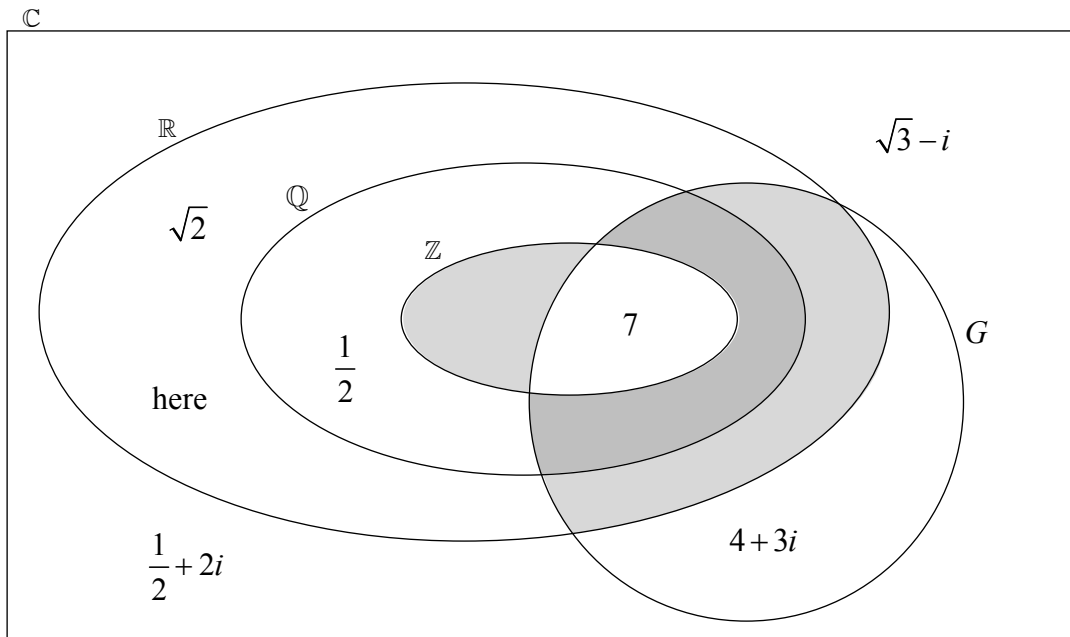
| | $x < 3$ | $3 < x < 5$ | $x > 5$ |
|-----------------------|---------|-------------|---------|
| $-x+5$ | + | + | - |
| $x-3$ | - | + | + |
| $\frac{-x+5}{2(x-3)}$ | - | + | - |

Question 2

(25 marks)

Let G be the set $\{x + yi \mid x, y \in \mathbb{Z}, i^2 = -1\}$.

Consider the Venn diagram below.



(a) There are three regions in the diagram that represent empty sets. One of these is shaded. Shade in the other two.

(b) Insert each of the following numbers in its correct region on the diagram.

$\sqrt{2}$

7

$\sqrt{3} - i$

$4 + 3i$

$\frac{1}{2}$

$\frac{1}{2} + 2i$

(c) Consider the product ab , where $a \in G$ and $b \in \mathbb{Q}$. There is a non-empty region in the diagram where ab cannot be. Write the word “here” in this region.

Real and imaginary parts of ab must both be rational, (rational \times integer). So ab cannot be in $\mathbb{R} \setminus \mathbb{Q}$.

Question 3**(25 marks)**

The complex number z has modulus $5\frac{1}{16}$ and argument $\frac{4\pi}{9}$.

- (a) Find, in polar form, the four complex fourth roots of z .
(That is, find the four values of w for which $w^4 = z$.)

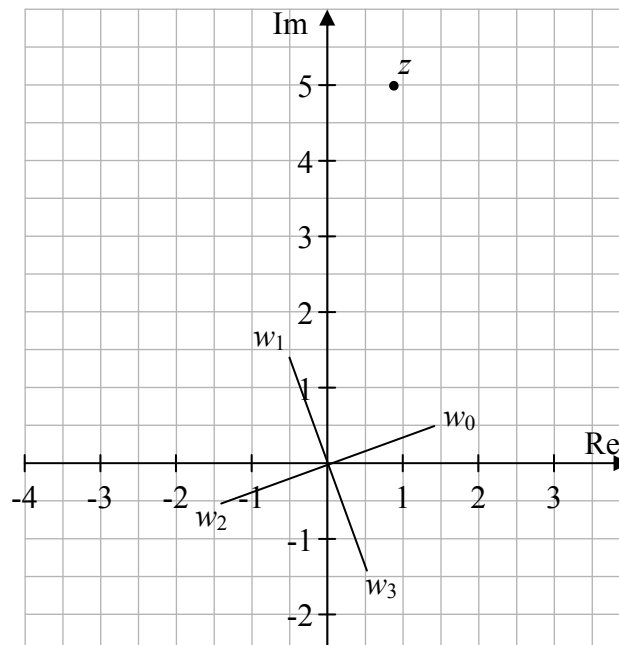
$$w^4 = \frac{81}{16} \left(\cos\left(\frac{4\pi}{9} + 2n\pi\right) + i \sin\left(\frac{4\pi}{9} + 2n\pi\right) \right)$$

$$w = \frac{3}{2} \left(\cos\left(\frac{\pi}{9} + \frac{n\pi}{2}\right) + i \sin\left(\frac{\pi}{9} + \frac{n\pi}{2}\right) \right), \quad n = 0, 1, 2, 3.$$

$$w = \frac{3}{2} \left(\cos\left(\frac{\pi}{9}\right) + i \sin\left(\frac{\pi}{9}\right) \right), \quad \frac{3}{2} \left(\cos\left(\frac{11\pi}{18}\right) + i \sin\left(\frac{11\pi}{18}\right) \right),$$

$$\frac{3}{2} \left(\cos\left(\frac{10\pi}{9}\right) + i \sin\left(\frac{10\pi}{9}\right) \right), \quad \frac{3}{2} \left(\cos\left(\frac{29\pi}{18}\right) + i \sin\left(\frac{29\pi}{18}\right) \right)$$

- (b) z is marked on the Argand diagram below.
On the same diagram, show the four answers to part (a).



Question 4**(25 marks)**

- (a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

$$P(n): \quad a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\text{Check } P(1): \quad a = \frac{a(1-r)}{1-r}, \text{ which is true.}$$

$$\text{Assume } P(k): \quad a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

$$\begin{aligned} \text{Then:} \quad & \underbrace{a + ar + ar^2 + \dots + ar^{k-1}} + ar^k \\ &= \frac{a(1-r^k)}{1-r} + ar^k \\ &= \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\ &= \frac{a(1-r^k + r^k - r^{k+1})}{1-r} \\ &= \frac{a(1-r^{k+1})}{1-r} \\ & \text{which establishes } P(k+1). \end{aligned}$$

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1))\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

- (b) By writing the recurring part as an infinite geometric series, express the following number as a fraction of integers:

$$5.\dot{2}\dot{1} = 5.2121212121\dots$$

$$\begin{aligned} 5.\dot{2}\dot{1} &= 5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \dots \\ &= 5 + [\text{geometric series with } a = \frac{21}{100}, \quad r = \frac{1}{100}]. \\ &= 5 + \frac{\frac{21}{100}}{1 - \frac{1}{100}} = 5 + \frac{21}{100-1} = 5\frac{21}{99} = 5\frac{7}{33}. \end{aligned}$$

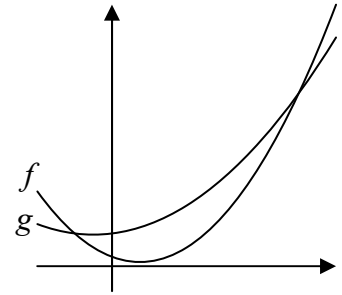
Question 5**(25 marks)**

The functions f and g are defined for $x \in \mathbb{R}$ as

$$f : x \mapsto 2x^2 - 3x + 2 \quad \text{and}$$

$$g : x \mapsto x^2 + x + 7.$$

- (a) Find the co-ordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.



$$f(x) = g(x)$$

$$2x^2 - 3x + 2 = x^2 + x + 7$$

$$x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) = 0$$

$$x = -1, \quad x = 5.$$

$$f(-1) = 7 \Rightarrow (-1, 7)$$

$$f(5) = 37 \Rightarrow (5, 37)$$

- (b) Find the area of the region enclosed between the two curves.

$$A = \int_{-1}^5 (g(x) - f(x)) dx$$

$$= \int_{-1}^5 (-x^2 + 4x + 5) dx$$

$$= \left[\frac{-x^3}{3} + 2x^2 + 5x \right]_{-1}^5$$

$$= \left(\frac{-125}{3} + 50 + 25 \right) - \left(\frac{1}{3} + 2 - 5 \right)$$

$$= 36.$$

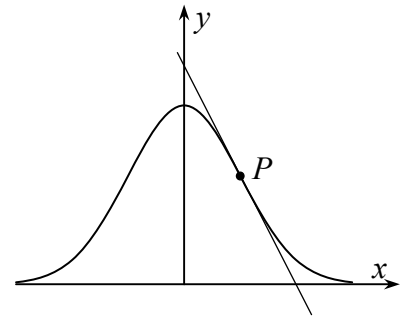
Question 6**(25 marks)**

(a) Let $f(x) = e^{-\frac{1}{2}x^2}$.

Show that the second derivative of $f(x)$ with respect to x is $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$.

$$\begin{aligned} f'(x) &= \underbrace{-x}_u \underbrace{e^{-\frac{1}{2}x^2}}_v \\ f''(x) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= -x \left(-xe^{-\frac{1}{2}x^2} \right) + e^{-\frac{1}{2}x^2} (-1) \\ &= (x^2 - 1)e^{-\frac{1}{2}x^2} \end{aligned}$$

- (b) The point P in the first quadrant is a point of inflection of the curve $y = e^{-\frac{1}{2}x^2}$.
Show that the tangent at P crosses the x -axis at $(2, 0)$.



Point of inflexion when $f''(x) = 0$.

$$(x^2 - 1)e^{-\frac{1}{2}x^2} = 0 \Rightarrow (x^2 - 1) = 0 \Rightarrow x = \pm 1.$$

First quadrant $\Rightarrow x = 1 \Rightarrow y = f(1) = e^{-\frac{1}{2}}$, and slope of tangent is $f'(1) = -e^{-\frac{1}{2}}$

So the equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1)$$

Crosses x -axis when $y = 0 \Rightarrow -e^{-\frac{1}{2}} = -e^{-\frac{1}{2}}(x - 1)$

$$1 = x - 1$$

$$x = 2$$

Answer **all three** questions from this section.

Question 7

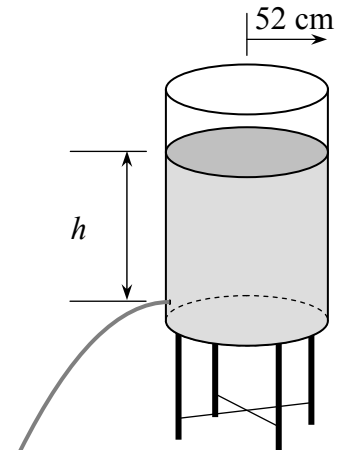
(50 marks)

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.

Over a certain 20-minute period, the height of the surface of the water is given by the formula

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole, and t is the time in seconds from a particular instant $t = 0$.



- (a) What is the height of the surface at time $t = 0$?

$$h(0) = 10^2 = 100 \text{ cm.}$$

- (b) After how many seconds will the height of the surface be 64 cm?

$$\begin{aligned} \left(10 - \frac{t}{200}\right)^2 &= 64 \\ 10 - \frac{t}{200} &= 8 \quad (\text{since } t > 0) \\ t &= 400 \end{aligned}$$

Answer: 400 seconds.

- (c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.

Give your answer correct to the nearest cm^3 per second.

$$V = \pi r^2 h = \pi(52)^2 h = 2704\pi h.$$

$$\frac{dV}{dt} = 2704\pi \frac{dh}{dt}$$

$$= 2704\pi \left(\frac{-2}{25}\right) = -216.32\pi$$

\therefore Volume is decreasing at $216.32\pi \text{ cm}^3 \text{ s}^{-1} \approx 680 \text{ cm}^3 \text{ s}^{-1}$.

$$\frac{dh}{dt} = 2 \left(10 - \frac{t}{200}\right) \frac{-1}{200}$$

$$\left. \frac{dh}{dt} \right|_{t=400} = \frac{-2}{25}$$

- (d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

$$\frac{dV}{dt} = Av$$

$$216 \cdot 32 \pi = \pi 1^2 v$$

$$v = 216 \cdot 32 \text{ cm s}^{-1}$$

- (e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of \sqrt{h} .

$$v = -\frac{1}{\pi} \frac{dV}{dt}$$

$$= -\frac{1}{\pi} 2704 \pi \frac{dh}{dt}$$

$$= 27 \cdot 04 \left(10 - \frac{t}{200} \right)$$

$$= 27 \cdot 04 \sqrt{h}$$

which is a constant multiple of \sqrt{h}

- (f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole. Find, correct to one decimal place, the value of c for this hole.

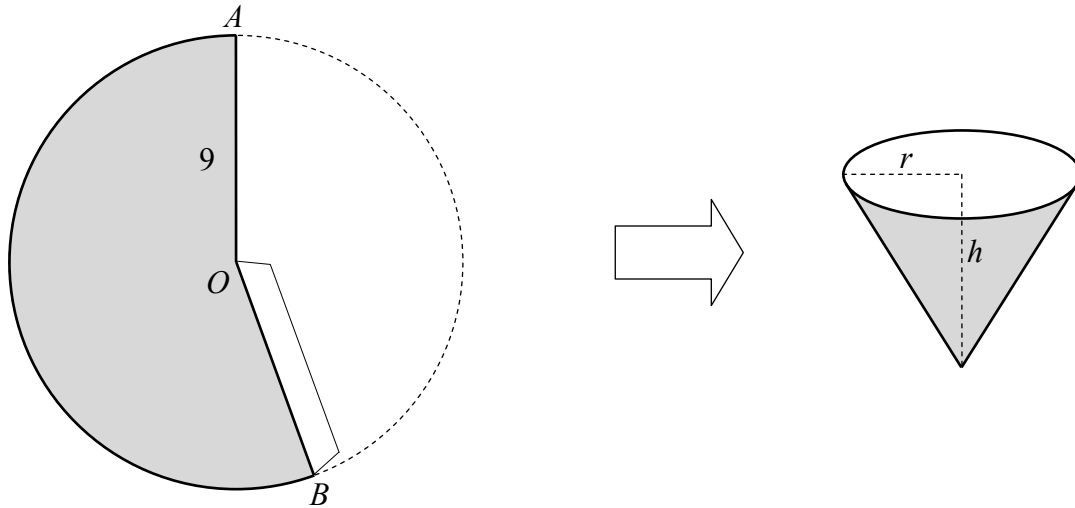
$$c\sqrt{1962} = 27 \cdot 04$$

$$c \approx 0 \cdot 6$$

Question 8**(50 marks)**

A company uses waterproof paper to make disposable conical drinking cups. To make each cup, a sector AOB is cut from a circular piece of paper of radius 9 cm. The edges AO and OB are then joined to form the cup, as shown.

The radius of the rim of the cup is r , and the height of the cup is h .



- (a) By expressing r^2 in terms of h , show that the capacity of the cup, in cm^3 , is given by the formula

$$V = \frac{\pi}{3}h(81 - h^2).$$

$$r^2 + h^2 = 9^2$$

$$r^2 = 81 - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi h}{3}(81 - h^2)$$

- (b) There are two positive values of h for which the capacity of the cup is $\frac{154\pi}{3}$.

One of these values is an integer.

Find the two values.

Give the non-integer value correct to two decimal places.

$$\frac{\pi h}{3}(81-h^2) = \frac{154\pi}{3}$$

$$h(81-h^2) = 154$$

$$h^3 - 81h + 154 = 0$$

Integer root is a factor of 154 $\Rightarrow \in \{1, 2, 7, 14, 11, 22, 77, 154\}$

$h = 1$ is not a solution; $h = 2$ is a solution.

$$\begin{array}{r} h^2 + 2h - 77 \\ h-2 \overline{) h^3 + 0h^2 - 81h + 154} \\ \underline{h^3 - 2h^2} \\ 2h^2 - 81h \\ \underline{2h^2 - 4h} \\ -77h + 154 \\ \underline{-77h + 154} \\ 0 \end{array}$$

$$h^2 + 2h - 77 = 0$$

$$(h+1)^2 - 78 = 0$$

$$h = -1 \pm \sqrt{78}$$

Positive solutions are $h = 2$, $h \approx 7.83$

- (c) Find the maximum possible volume of the cup, correct to the nearest cm^3 .

$$V = \frac{\pi h}{3}(81 - h^2), \quad h \in [0, 9]$$

$$= \frac{\pi}{3}(81h - h^3)$$

$$\frac{dV}{dh} = \pi(27 - h^2)$$

Local max/min when $\frac{dV}{dh} = 0 \Rightarrow h = \sqrt{27}$. (Clearly a max., since $V(0) = V(9) = 0$.)

$$V_{\max} = \pi(27\sqrt{27} - 9\sqrt{27}) = 18\sqrt{27}\pi \approx 294 \text{ cm}^3$$

- (d) Complete the table below to show the radius, height, and capacity of each of the cups involved in parts (b) and (c) above. In each case, give the radius and height correct to two decimal places.

| | cups in part (b) | | cup in part (c) |
|------------------|---|---|--------------------|
| radius (r) | 8.77 cm | 4.43 cm | 7.35 cm |
| height (h) | 2 cm | 7.83 cm | 5.20 cm |
| capacity (V) | $\frac{154\pi}{3} \approx 161 \text{ cm}^3$ | $\frac{154\pi}{3} \approx 161 \text{ cm}^3$ | 294 cm^3 |

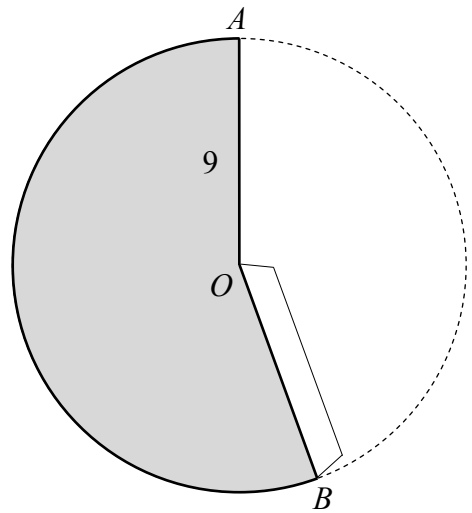
- (e) In practice, which one of the three cups above is the most reasonable shape for a conical cup? Give a reason for your answer.

The middle one (radius 4.43 cm, height 7.83 cm).
The others are much too wide and shallow to hold.

- (f) For the cup you have chosen in part (e), find the measure of the angle AOB that must be cut from the circular disc in order to make the cup.
Give your answer in degrees, correct to the nearest degree.

Circumference of rim = $2\pi r \approx 8.86\pi \approx 27.86$ cm.

$$\theta = \frac{l}{r} = \frac{27.86}{9} = 3.096 \text{ rad} \approx 177^\circ$$



Question 9

(50 marks)

The *atmospheric pressure* is the pressure exerted by the air in the earth’s atmosphere. It can be measured in kilopascals (kPa). The average atmospheric pressure varies with altitude: the higher up you go, the lower the pressure is.

Some students are investigating this variation in pressure, using some data that they found on the internet. They have information about the average pressure at various altitudes.

Six of the entries in the data set are as shown in the table below:

| | | | | | | |
|----------------|-------|------|------|------|------|------|
| altitude (km) | 0 | 1 | 2 | 3 | 4 | 5 |
| pressure (kPa) | 101.3 | 89.9 | 79.5 | 70.1 | 61.6 | 54.0 |

By looking at the pattern, the students are trying to find a suitable model to match the data.

(a) Hannah suggests that this is approximately a geometric sequence. She says she can match the data fairly well by taking the first term as 101.3 and the common ratio as 0.883.

(i) Complete the table below to show the values given by Hannah’s model, correct to one decimal place.

| | | | | | | |
|----------------|-------|-------|-------|-------|------|-------|
| altitude (km) | 0 | 1 | 2 | 3 | 4 | 5 |
| pressure (kPa) | 101.3 | 89.4 | 79.0 | 69.7 | 61.6 | 54.4 |
| | | -0.6% | -0.6% | -0.6% | 0% | +0.7% |

(ii) By considering the percentage errors in the above values, insert an appropriate number to complete the statement below.

“Hannah’s model is accurate to within 1 %.”

(b) Thomas suggests modelling the data with the following exponential function:

$$p = 101.3 \times e^{-0.1244h}$$

where p is the pressure in kilopascals, and h is the altitude in kilometres.

(i) Taking any **one** value other than 0 for the altitude, verify that the pressure given by Thomas’s model and the pressure given by Hannah’s model differ by less than 0.01 kPa.

| | |
|-------------|--|
| Altitude 1: | $101.3e^{-0.1244} - 101.3(0.883) \approx 0.0027 < 0.01$ |
| Altitude 2: | $101.3e^{-0.1244 \times 2} - 101.3(0.883)^2 \approx 0.0048 < 0.01$ |
| Altitude 3: | $101.3e^{-0.1244 \times 3} - 101.3(0.883)^3 \approx 0.0063 < 0.01$ |
| Altitude 4: | $101.3e^{-0.1244 \times 4} - 101.3(0.883)^4 \approx 0.0074 < 0.01$ |
| Altitude 5: | $101.3e^{-0.1244 \times 5} - 101.3(0.883)^5 \approx 0.0081 < 0.01$ |

- (ii) Explain how Thomas might have arrived at the value of the constant 0.1244 in his model.

He might have assumed it was of the form $101.3e^{-kt}$ and then used one of the observations to find k .

He might have put various values of t into $p(t) = 101.3e^{-kt}$, and found an average of the resulting values of k .

He might have got the natural log of the ratio of consecutive terms.

He might have plotted the log of the pressure against the altitude and used the slope of the best-fit line to find k .

- (c) Hannah's model is *discrete*, while Thomas's is *continuous*.

- (i) Explain what this means.

Hannah's model gives values for the pressure at separate (whole number) values for the altitude.

Thomas's model gives a value for the pressure at any real value of the altitude, whether it's a whole number or not.

- (ii) State one advantage of a continuous model over a discrete one.

You are not restricted to the specific discrete values of the independent variable; you can also work with values between any two given values – any value you like.

- (d) Use Thomas's model to estimate the atmospheric pressure at the altitude of the top of Mount Everest: 8848 metres.

$$p(8848) = 101.3e^{-0.1244 \times 8848} \approx 33.7 \text{ kPa} .$$

- (e) Using Thomas's model, find an estimate for the altitude at which the atmospheric pressure is half of its value at sea level (altitude 0 km).

$$\begin{aligned}p(h) &= \frac{1}{2} p(0) \\101.3e^{-0.1244h} &= \frac{1}{2}(101.3) \\e^{0.1244h} &= 2 \\0.1244h &= \ln 2 \\h &\approx 5.57 \text{ km}\end{aligned}$$

- (f) People sometimes experience a sensation in their ears when the pressure changes. This can happen when travelling in a fast lift in a tall building. Experiments indicate that many people feel such a sensation if the pressure changes rapidly by 1 kilopascal or more. Suppose that such a person steps into a lift that is close to sea level. Taking a suitable approximation for the distance between two floors, estimate the number of floors that the person would need to travel in order to feel this sensation.

$$\begin{aligned}p(h) &= 100.3 \\101.3e^{-0.1244h} &= 100.3 \\e^{-0.1244h} &= \frac{100.3}{101.3} \\-0.1244h &= \ln\left(\frac{100.3}{101.3}\right) \\h &= \frac{1}{0.1244}(\ln 101.3 - \ln 100.3) \\h &\approx 0.0797 \text{ km} \approx 80 \text{ m}\end{aligned}$$

Allowing 3 m per floor, it's about 27 floors.

Marking Scheme – Paper 1

Structure of the marking scheme

Candidate responses are marked according to different scales, depending on the types of response anticipated. Scales labelled A divide candidate responses into two categories (correct and incorrect). Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on. The scales and the marks that they generate are summarised in this table:

| Scale label | A | B | C | D | E |
|------------------|---|-----------|---------------|------------------|---|
| No of categories | 2 | 3 | 4 | 5 | 5 |
| 5 mark scale | | 0, 3, 5 | 0, 3, 4, 5 | | |
| 10 mark scale | | 0, 7, 10 | 0, 5, 8, 10 | 0, 3, 5, 8, 10 | |
| 15 mark scale | | 0, 11, 15 | 0, 8, 14, 15 | 0, 5, 10, 14, 15 | |
| 20 mark scale | | | 0, 14, 18, 20 | 0, 6, 12, 18, 20 | |

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)

- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)

- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

E-scales (six categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response almost half-right (lower middle partial credit)
- response more than half-right (upper middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving incorrect rounding or omission of units, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk. Thus, for example, *scale 10C** indicates that 9 marks may be awarded.

Summary of mark allocations and scales to be applied

Section A

Question 1

- (a) 15C
- (b) 10D

Question 2

- (a) 15B
- (b) 5C
- (c) 5B

Question 3

- (a) 20C
- (b) 5B

Question 4

- (a) 10C
- (b) 15C

Question 5

- (a) 10C
- (b) 15D

Question 6

- (a) f' 10B
- (a) f'' 10B
- (b) 5C

Section B

Question 7

- (a) 15B*
- (b) 15C*
- (c) 5C*
- (d) 10B*
- (e)&(f) 5C

Question 8

- (a) 10B
- (b) 20D*
- (c) 5C*
- (d) 5C*
- (e) 5B
- (f) 5C*

Question 9

- (a)(i) 10C*
- (a)(ii) 5B
- (b)(i) 5B
- (b)(ii) 5B
- (c)(i)&(ii) 5B
- (d) 10B*
- (e) 5C
- (f) 5C

Detailed marking notes

Section A

Question 1

(a) Scale 15C (0, 8, 14, 15).

Low partial credit:

- * Any attempt at trial and error
- * Writes a in terms of b , or b in terms of a , and stops.
- * Any reasonable first step

High partial credit:

- * Solves for one variable (two values), more or less correctly
- * Substitutes $-2b - 1$ for a in the non-linear equation and some further progress
- * Substitutes $\frac{-a-1}{2}$ for b in the non-linear equation and some further progress.

(b) Scale 10D (0, 3, 5, 8, 10).

Low partial credit:

- * Any reasonable first step.

Mid partial credit:

- * Finds particular values of x for which the inequality holds

High partial credit:

- * Solves the relevant quadratic to find the solutions, $x = 3$ or $x = 5$
- * Deals more or less correctly with one case only ($x < 3$ or $x \geq 5$)
- * Solution shown on a graph
- * Substantial progress towards full solution

Question 2

(a) Scale 15B (0, 11, 15).

Partial credit:

- * One correct region shaded
- * Two regions containing no elements shaded

(b) Scale 5C (0, 3, 4, 5).

Low partial credit:

- * At least one correct.

High partial credit:

- * Three, four or five correct.

Note: any duplicate entry counts as incorrect.

(c) Scale 5B (0, 3, 5).

No credit:

- * Answer in the wrong place, without explanation or work.

Partial credit:

- * Takes any element of G and any element of Q and finds their product
- * Some relevant work or explanation of merit.

- * Writes $\frac{p}{q}(x + iy)$

Full credit:

- * Answer in correct place, with or without explanation or work.

Question 3

(a) Scale 20C (0, 14, 18, 20).

Low partial credit:

* Any reasonable first step

* $\cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9}$ or $\cos 80^\circ + i \sin 80^\circ$ appears without any further work

High partial credit:

* Correct solution with one or two minor errors

* One fully correct value

* $\left(5 \frac{1}{16}\right)^{\frac{1}{4}} \left(\cos \frac{1}{4} \left(\frac{4\pi}{9} \right) + i \sin \frac{1}{4} \left(\frac{4\pi}{9} \right) \right)$

* Not in polar form

(b) Scale 5B (0, 3, 5).

Partial credit:

* Some work of merit.

* If moduli are not the same, then partial credit only

Note: need four reasonable solutions to part (a) in order to gain full credit in part (b).

Question 4

(a) Scale 10C (0, 5, 8, 10).

Low partial credit:

- * Any reasonable first step
- * Correct proof without using induction

High partial credit:

- * Solution mostly correct, including evidence of understanding the logic of proof by induction.
- * Fully correct $P(k) \Rightarrow P(k+1)$.

(b) Scale 15C (0, 8, 14, 15).

Low partial credit:

- * $S_{\infty} = \frac{a}{1-r}$

- * Correct answer without work
- * Any reasonable first step

High partial credit:

- * $5 + \frac{21}{100} + \frac{21}{10000} + \frac{21}{1000000} + \dots$

- * Correct solution with minor arithmetic errors.

Question 5

(a) Scale 10C (0, 5, 8, 10).

Low partial credit:

- * Constructs a table of values but fails to finish
- * Any reasonable first step

High partial credit:

- * Correct x -values but no y -values (or incorrect y -values).
- * Solution with one minor error.

(b) Scale 15D (0, 5, 10, 14, 15).

Low partial credit:

- * Area enclosed = Area under g – Area under f .
- * Shade the enclosed area
- * Indicate the limits on the graph
- * Any reasonable first step

Mid partial credit:

- * Correct integration of $f(x)$ or $g(x)$

High partial credit:

- * Correct integration of $(f(x) - g(x))$ or $(g(x) - f(x))$, or both $f(x)$ and $g(x)$.
- * Correct solution with minor arithmetic errors, and/or mishandles negative integral.

Question 6

(a) First derivative: Scale 10B (0, 7, 10).

Partial credit:

* Any reasonable first step

Second derivative: Scale 10B (0, 7, 10).

Partial credit:

* Any reasonable first step

(b) Scale 5C (0, 3, 4, 5).

Low partial credit:

* Any reasonable first step

* $y - 0 = m(x - 2)$

* States $f''(x) = 0$ and stops.

High partial credit:

* Full solution with one or two errors.

* Correct tangent and stops.

Section B

Question 7

- (a) Scale 15B* (0, 11, [14], 15).

Partial credit:

- * Any reasonable first step, such as any attempt to find $h(0)$.

- (b) Scale 15C* (0, 8, 14, [14], 15).

Low partial credit:

- * Any reasonable first step, such as setting $h(t) = 64$
- * Attempts a systematic search (trial and improvement)
- * Attempts graphical solution

High partial credit:

- * Solution with one or two errors

Full Credit:

- * Correct answer by trial & improvement
- * Correct answer by graphical method.

- (c) Scale 5C* (0, 3, 4, [4], 5).

Low partial credit:

- * Assumes the rate of decrease is linear i.e. finds rate of decrease from a volume of $270,400\pi \text{ cm}^3$ to a volume of $173,056\pi \text{ cm}^3$.
- * $\frac{dv}{dt} = \frac{dh}{dt} \cdot \frac{dv}{dh}$
- * Any reasonable first step.

High partial credit:

- * Finds $\frac{dh}{dt}$ at $t = 400$, or comparable relevant work.
- * Solution by correct method, but with some errors.

- (d) Scale 10B* (0, 7, [9], 5).

Partial credit:

- * Any reasonable first step.
- * $A = \pi r^2$

Note: Ignore minus sign.

- (e)&(f) Scale 5C (0, 3, 4, 5).

Low partial credit:

- * Any work of merit in either (e) or (f).

High partial credit:

- * Work of merit in both (e) and (f)
- * One fully correct solution in either (e) or (f)

Question 8

(a) Scale 10B (0, 7, 10).

Partial credit:

- * $r^2 = 81 - h^2$ and stops
- * Any reasonable first step

(b) Scale 20D* (0, 6, 12, 18, [19] 20).

Low partial credit:

- * Any reasonable first step.

Mid-partial credit:

- * Any step beyond $\frac{154\pi}{3} = \frac{\pi}{3}h(81 - h^2)$

High partial credit:

- * Substantial correct progress made (e.g. 1 root found + some further correct progress)
- * Solution by correct method, but with some errors.

(c) Scale 5C* (0, 3, 4, [4], 5).

Low partial credit:

- * Any reasonable first step.

High partial credit:

- * Finds h correctly and stops.
- * Finds $\frac{dV}{dh}$ correctly and sets $\frac{dV}{dh} = 0$
- * Error(s) in derivative, and finishes correctly or nearly correctly

(d) Scale 5C* (0, 3, 4, [4], 5).

Low partial credit:

- * Any one correct entry
- * Any correct relevant work

High partial credit:

- * Five or six correct entries (based on previous work)

(e) Scale 5B (0, 3, 5).

No credit:

- * Incorrect answer, with or without explanation.

Partial credit:

- * Correct answer, without explanation.
- * Cup with maximum volume.

(f) Scale 5C* (0, 3, 4, [4], 5).

Low partial credit:

* Reasonable work with any one of $A = \pi r^2$, $C = 2\pi r$, $A = \pi rl$, $A = \frac{1}{2}r^2\theta$

* Any reasonable first step

High partial credit:

* Correct answer in radians

Question 9

(a)(i) Scale 10C* (0, 5, 8, [9] 10).

Low partial credit:

* One correct entry (i.e. one entry with correct ratio to previous entry)

High partial credit:

* Four correct entries

(a)(ii) Scale 5B (0, 3, 5).

Partial credit:

* Any correct relevant work

* Answer outside the range [0.5%, 2%]

(b)(i) Scale 5B (0, 3, 5).

Partial credit:

* Any correct relevant work

(b)(ii) Scale 5B (0, 3, 5).

Partial credit:

* Incomplete explanation, such as “solved an equation”, “used a graph”, “used two consecutive terms”

* “He saw that 101.3 kPa was the pressure at sea level.”

(c) Scale 5B (0, 3, 5).

* Mark both parts together, as there may be overlap between the answers: the explanation in (i) may make the advantage clear, or the advantage in (ii) might clearly imply an understanding of the meaning.

Partial credit:

* Answer displays partial understanding of concepts and/or advantage. e.g.:

“The first is for natural numbers; the second is for real numbers”. (No indication of advantage.)

“You can count discrete things but not continuous things”

* Concepts understood but reversed

Full credit:

* Answer displays good understanding of concepts and advantage

(d) Scale 10B* (0, 7, [9], 10).

Partial credit:

* Any substitution into formula.

* Any correct relevant work

Note: Accept answer rounded to nearest whole number, or to 1 or 2 decimal places. Otherwise, rounding penalty applies.

(e) Scale 5C* (0, 3, 4, [4], 5).

Low partial credit:

* Any reasonable first step

High partial credit:

* Solution with error(s), subject to including a correct relevant conversion from indices to logs or vice versa.

Note: Accept answer rounded to 1, 2 or 3 decimal places. Otherwise, rounding penalty applies.

(f) Scale 5C* (0, 3, 4, [4], 10).

Low partial credit:

* Any reasonable first step

Mid partial credit:

Solution with error(s), subject to including a correct relevant conversion from indices to logs or vice versa.

High partial credit:

* Solves for h correctly, or nearly correctly, but then omits conversion to floors or uses estimate outside the interval $[2, 5]$



Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination, 2012

Mathematics
(Project Maths – Phase 1)

Paper 2

Higher Level

Monday 11 June Morning 9:30 – 12:00

300 marks

Model Solutions – Paper 2

Note: the model solutions for each question are not intended to be exhaustive – there may be other correct solutions. Any examiner unsure of the validity of the approach adopted by a particular candidate to a particular question should contact his/her advising examiner.

Instructions

There are **two** sections in this examination paper.

| | | | |
|-----------|---------------------------|-----------|-------------|
| Section A | Concepts and Skills | 150 marks | 6 questions |
| Section B | Contexts and Applications | 150 marks | 2 questions |

Answer **all eight** questions, as follows:

In Section A, answer:

Questions 1 to 5 and
either Question 6A **or** Question 6B.

In Section B, answer Question 7 and Question 8.

Write your answers in the spaces provided in this booklet. You will lose marks if you do not do so. There is space for extra work at the back of the booklet. You may also ask the superintendent for more paper. Label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Answer **all six** questions from this section.

Question 1

(25 marks)

- (a) Given the co-ordinates of the vertices of a quadrilateral $ABCD$, describe **three** different ways to determine, using co-ordinate geometry techniques, whether the quadrilateral is a parallelogram.

1. Check whether both pairs of opposite sides have the same slope (slope formula).
2. Check whether both pairs of opposite sides are equal in length (distance formula).
3. Check whether the midpoints of the diagonals coincide (diagonals bisecting each other).
4. Check whether the translation from A to B is the same as the translation from D to C [or equivalent.]
5. Check whether a pair of opposite sides have the same slope and are equal in length (slope and distance formulae).
6. Use slopes and the formula for the angle between two lines to check whether both pairs of opposite angles are equal.
7. Use slopes and the formula for the angle between two lines to check whether $|\angle A| + |\angle B| = 180^\circ$, and $|\angle C| + |\angle D| = 180^\circ$. [or equivalent]

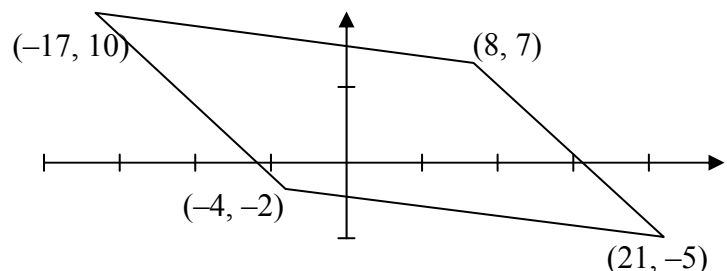
- (b) Using **one** of the methods you described, determine whether the quadrilateral with vertices $(-4, -2)$, $(21, -5)$, $(8, 7)$ and $(-17, 10)$ is a parallelogram.

Midpoints of diagonals:

$$\left(\frac{-4+8}{2}, \frac{-2+7}{2} \right) = \left(2, \frac{5}{2} \right)$$

$$\left(\frac{-17+21}{2}, \frac{10-5}{2} \right) = \left(2, \frac{5}{2} \right)$$

Equal \Rightarrow parallelogram.



For other methods: slopes are $-\frac{12}{13}$ and $\frac{3}{25}$; side-lengths are $\sqrt{313}$ and $\sqrt{634}$, translations are $(x, y) \rightarrow (x+25, y-3)$ and $(x, y) \rightarrow (x+13, y-12)$, or reverse.

Question 2**(25 marks)**

The equations of two circles are:

$$c_1 : x^2 + y^2 - 6x - 10y + 29 = 0$$

$$c_2 : x^2 + y^2 - 2x - 2y - 43 = 0$$

(a) Write down the centre and radius-length of each circle.

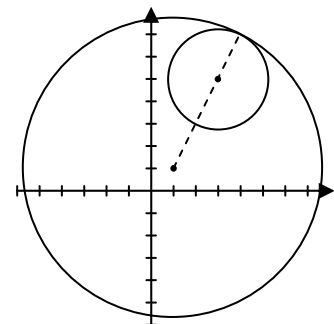
$$c_1 : (x-3)^2 + (y-5)^2 = 5$$

 \therefore centre (3, 5); radius $\sqrt{5}$.

$$c_2 : (x-1)^2 + (y-1)^2 = 45$$

 \therefore centre (1, 1); radius $\sqrt{45} = 3\sqrt{5}$.**(b)** Prove that the circles are touching.

$$\text{Distance between centres: } \sqrt{(3-1)^2 + (5-1)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

The distance between the centres is the difference of the radii \Rightarrow circles touch (internally).

- (c) Verify that (4, 7) is the point that they have in common.

$$4^2 + 7^2 - 6(4) - 10(7) + 29 = 0 \Rightarrow (4, 7) \in c_1$$

$$4^2 + 7^2 - 2(4) - 2(7) - 43 = 0 \Rightarrow (4, 7) \in c_2$$

OR

$$c_1 - c_2: x + 2y - 18 = 0 \Rightarrow x = -2y + 18$$

$$(-2y + 18)^2 + y^2 - 6(-2y + 18) - 10y + 29 = 0$$

$$(y - 7)^2 = 0$$

$$y = 7$$

$$x = 4$$

$\therefore (4, 7)$ common

- (d) Find the equation of the common tangent.

Slope from (3, 5) to (4, 7) is: $\frac{7-5}{4-3} = 2$

\therefore slope of tangent = $-\frac{1}{2}$.

Equation of tangent:

$$y - 7 = -\frac{1}{2}(x - 4)$$

$$2y - 14 = -x + 4$$

$$x + 2y - 18 = 0$$

OR

Equation of Tangent: $c_1 - c_2: x + 2y - 18 = 0$

OR

$$(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2$$

$$(x - 3)(4 - 3) + (y - 5)(7 - 5) = (\sqrt{5})^2$$

$$(x - 3) + (y - 5)(2) = 5$$

$$x + 2y - 18 = 0$$

OR

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$4x + 7y - 3(x + 4) - 5(y + 7) + 29 = 0$$

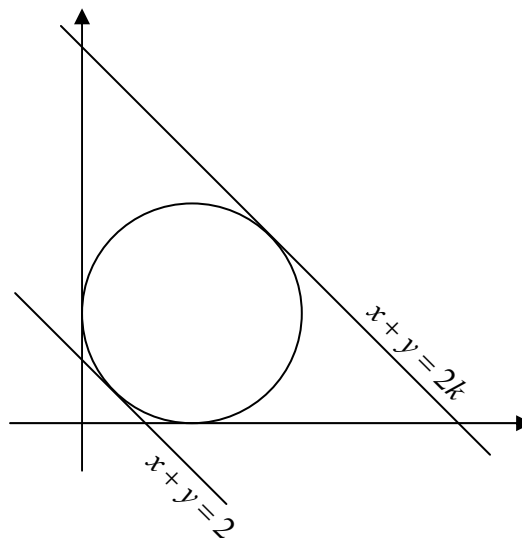
$$x + 2y - 18 = 0$$

Question 3

(25 marks)

The circle shown in the diagram has, as tangents, the x -axis, the y -axis, the line $x + y = 2$ and the line $x + y = 2k$, where $k > 1$.

Find the value of k .



$$r^2 + r^2 = (r + \sqrt{2})^2$$

$$2r^2 = r^2 + 2\sqrt{2}r + 2$$

$$r^2 - 2\sqrt{2}r - 2 = 0$$

$$(r - \sqrt{2})^2 = 4$$

$$r = \sqrt{2} + 2, \quad (r > 0)$$

(r, r) is midpoint of segment from $(1, 1)$ to (k, k) .

$$\frac{k+1}{2} = r$$

$$k = 2r - 1$$

$$k = 3 + 2\sqrt{2}$$

OR

$$\text{Equation of circle: } (x-r)^2 + (y-r)^2 = r^2$$

The line $x + y = 2$ intersects the circle at one point only.

$$y = 2 - x \Rightarrow (x-r)^2 + ((2-x)-r)^2 = r^2$$

$$\Rightarrow x^2 + (2-x)^2 + r^2 - 4r = 0$$

$$\Rightarrow 2x^2 - 4x + (r^2 - 4r + 4) = 0$$

One real root $\Rightarrow b^2 - 4ac = 0$

$$\Rightarrow 16 - 4(2)(r^2 - 4r + 4) = 0$$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

But $2 - \sqrt{2}$ is too small, so $r = 2 + \sqrt{2}$

$$(1, 1) \rightarrow (2 + \sqrt{2}, 2 + \sqrt{2}) \rightarrow (3 + 2\sqrt{2}, 3 + 2\sqrt{2}) = (k, k)$$

OR

Centre (r, r)

Perpendicular distance to $x + y - 2 = 0$ equals radius, r .

$$\left| \frac{r+r-2}{\sqrt{2}} \right| = r$$

$$\Rightarrow 2r - 2 = \pm r\sqrt{2}$$

$$r = \frac{2}{2 \pm \sqrt{2}} = 2 \mp \sqrt{2}$$

But $2 - \sqrt{2}$ is too small, so $r = 2 + \sqrt{2}$

$$(1, 1) \rightarrow (2 + \sqrt{2}, 2 + \sqrt{2}) \rightarrow (3 + 2\sqrt{2}, 3 + 2\sqrt{2}) = (k, k)$$

OR

Having found r as above, find k by setting perpendicular distance from centre (r, r) to $x + y - 2k = 0$ equal to r :

$$\left| \frac{r+r-2k}{\sqrt{2}} \right| = r$$

$$\Rightarrow 2r - 2k = \pm \sqrt{2}r$$

$$\Rightarrow 2(2 + \sqrt{2}) - 2k = \pm \sqrt{2}(2 + \sqrt{2})$$

$$\Rightarrow 4 + 2\sqrt{2} - 2k = \pm (2\sqrt{2} + 2)$$

$$\Rightarrow 2k = 4 + 2\sqrt{2} \pm (2\sqrt{2} + 2)$$

$$\Rightarrow k = 2 + \sqrt{2} \pm (\sqrt{2} + 1)$$

$$\Rightarrow k = 3 + 2\sqrt{2} \text{ or } 1.$$

$k = 1$ corresponds to the lower line, so the answer is $k = 3 + 2\sqrt{2}$

Question 4**(25 marks)**

A certain basketball player scores 60% of the free-throw shots she attempts. During a particular game, she gets six free throws.

- (a) What assumption(s) must be made in order to regard this as a sequence of Bernoulli trials?

Trials are independent of each other.
Probability of success is the same each time.

[Only two outcomes (Given)]
[Finite number of throws..... (Given)]

- (b) Based on such assumption(s), find, correct to three decimal places, the probability that:

- (i) she scores on exactly four of the six shots

$$P(X = 4) = {}^6C_4 (0.6)^4 (0.4)^2 = 0.31104 \\ = 0.311 \text{ to three decimal places.}$$

- (ii) she scores for the second time on the fifth shot.

Exactly one success among first four throws, followed by success on fifth:

$$({}^4C_1 (0.6)(0.4)^3)(0.6) = 0.09216 \\ = 0.092 \text{ to three decimal places.}$$

Question 5**(25 marks)**

A company produces calculator batteries. The diameter of the batteries is supposed to be 20 mm. The tolerance is 0.25 mm. Any batteries outside this tolerance are rejected. You may assume that this is the only reason for rejecting the batteries.



- (a) The company has a machine that produces batteries with diameters that are normally distributed with mean 20 mm and standard deviation 0.1 mm. Out of every 10 000 batteries produced by this machine, how many, on average, are rejected?

$$Z = \frac{20.25 - 20}{0.1} = 2.5$$

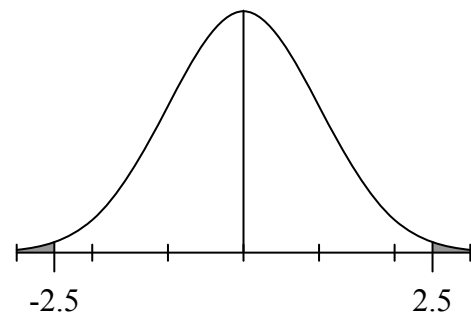
$$P(|X - 20| > 0.25) = P(|Z| > 2.5)$$

$$= 2(1 - P(Z \leq 2.5))$$

$$= 2(1 - 0.9938)$$

$$= 0.0124$$

$$\text{Answer} = 10\,000 \times 0.0124 = 124.$$



- (b) A setting on the machine slips, so that the mean diameter of the batteries increases to 20.05 mm, while the standard deviation remains unchanged. Find the percentage increase in the rejection rate for batteries from this machine.

$$P(X \leq 19.75) + P(X \geq 20.25) = P\left[Z \leq \frac{19.75 - 20.05}{0.1}\right] + P\left[Z \geq \frac{20.25 - 20.05}{0.1}\right]$$

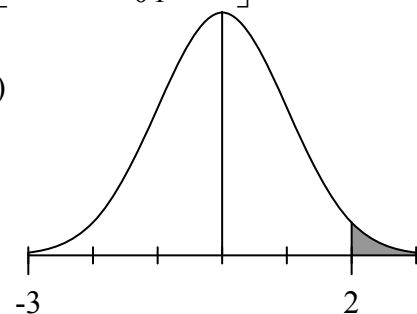
$$= P(Z \leq -3) + P(Z \geq 2)$$

$$= 1 - P(Z \leq 3) + 1 - P(Z \leq 2)$$

$$= 1 - 0.9987 + 1 - 0.9772$$

$$= 2 - 1.9759$$

$$= 0.0241$$



$$\frac{0.0241}{0.0124} = 1.9435\dots \Rightarrow 94.35\% \text{ increase}$$

$$\text{or increase: } 0.0241 - 0.0124 = 0.0117$$

$$\% \text{ Increase: } \left(\frac{0.0117}{0.0124}\right) 100 = 94.35\%$$

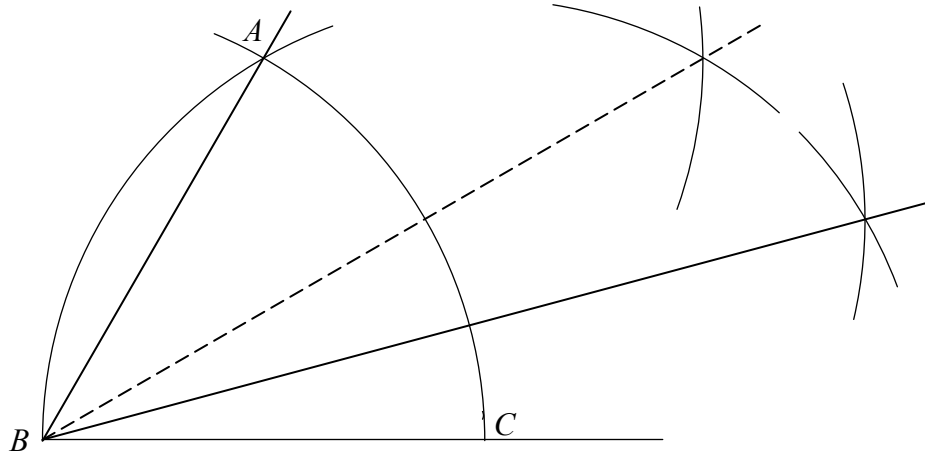
Question 6

(25 marks)

Answer **either** 6A **or** 6B.

Question 6A

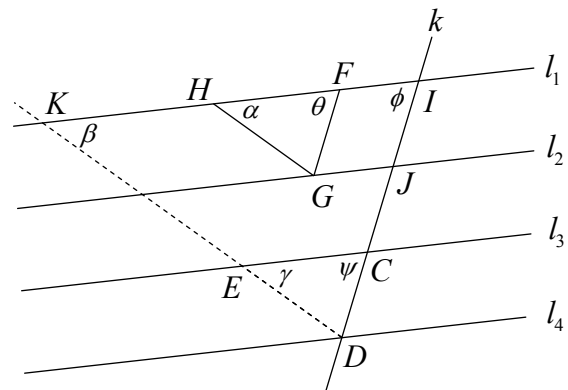
- (a) (i) Given the points B and C below, construct, without using a protractor or setsquare, a point A such that $|\angle ABC| = 60^\circ$.



- (ii) Hence construct, on the same diagram above, and using a compass and straight edge only, an angle of 15° .

Bisect 60° to get 30° ; bisect again to get 15° (as shown above)
 OR
 Construct a right angle and use it to construct 45° and combine with 60° to get 15° .

- (b) In the diagram, $l_1, l_2, l_3,$ and l_4 are parallel lines that make intercepts of equal length on the transversal k . FG is parallel to k , and HG is parallel to ED .



Prove that the triangles $\triangle CDE$ and $\triangle FGH$ are congruent.

$|CD| = |IJ|$ (given)
 $= |FG|$ (opposite sides of parallelogram)
 $\theta = \phi = \psi$ (corresponding angles)
 $\alpha = \beta = \gamma$ (corresponding angles)
 $\Rightarrow |\angle HGF| = |\angle EDC|$
 $\therefore \triangle CDE \equiv \triangle FGH$ (ASA)

OR

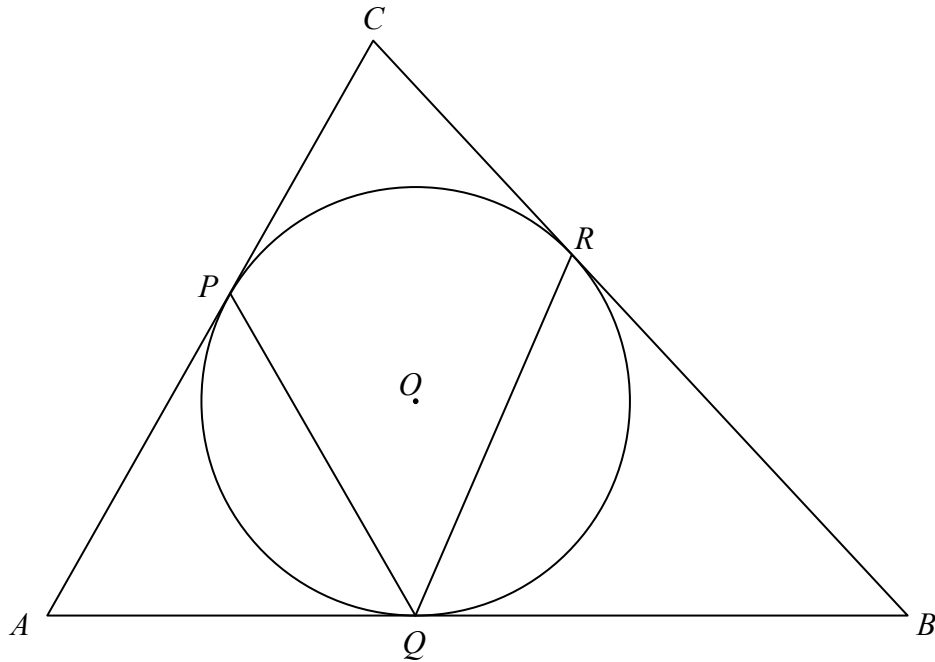
| | |
|---|--|
| $ CD = IJ $ (given) $= FG $ (opposite sides of parallelogram) $\theta = \phi = \psi$ (corresponding angles) $\alpha = \beta = \gamma$ (corresponding angles) $\therefore \triangle CDE \equiv \triangle FGH$ (ASA) | |
|---|--|

OR

Question 6B

The incircle of the triangle ABC has centre O and touches the sides at P, Q and R , as shown.

Prove that $|\angle PQR| = \frac{1}{2}(|\angle CAB| + |\angle CBA|)$.



| |
|--|
| $ \angle OQA = \angle OPA = 90^\circ$ (radius \perp tangent) $\therefore O, Q, A, P$ are concyclic. $ \angle OQP = \angle OAP $ (standing on same arc OP) $= \frac{1}{2} \angle PAQ $ (since AO is the bisector of $\angle PAQ$) Similarly, $ \angle OQR = \frac{1}{2} \angle QBR $ Adding these two gives the required result. |
|--|

OR

$$\begin{aligned}
|\angle OPC| &= |\angle ORC| = 90^\circ && \text{(radius } \perp \text{ tangent)} \\
\therefore |\angle PBR| &= 180^\circ - |\angle POR| && \text{(angles in any quadrilateral add up to } 360^\circ) \\
\text{But } |\angle PBR| &= 180^\circ - (|\angle CAB| + |\angle CBA|) && \text{(angles in a triangle)} \\
\text{So } |\angle POR| &= |\angle CAB| + |\angle CBA| \\
\text{But } |\angle PQR| &= \frac{1}{2} |\angle POR| \\
\text{So } |\angle PQR| &= \frac{1}{2} (|\angle CAB| + |\angle CBA|)
\end{aligned}$$

OR

$$\begin{aligned}
\text{Let } OA \cap PQ &= \{D\} \\
|OP| &= |OQ| \Rightarrow |AP| = |AQ| && \text{(Pythagoras)} \\
|\angle PAD| &= |\angle QAD| && \text{(bisector)} \\
\therefore \triangle PDA &\equiv \triangle QDA && \text{(S.A.S.)} \\
\therefore |\angle PDA| &= |\angle QDA| = 90^\circ \\
|\angle DAQ| &= 90^\circ - |\angle DQA| \\
&= |\angle OQD| \\
\therefore |\angle PAQ| &= 2|\angle OQD| \\
\text{Similarly, } |\angle RBQ| &= 2|\angle OQR| \\
\text{Adding these two gives the required result.}
\end{aligned}$$

OR

$$\begin{aligned}
\text{Let } OA \cap PQ &= \{D\} \\
|OP| &= |OQ| \Rightarrow |AP| = |AQ| && \text{(Pythagoras)} \\
|\angle APQ| &= |\angle AQP| && \text{(isosceles triangle theorem)} \\
\text{Similarly, } |\angle RQB| &= |\angle RBQ| \\
|\angle AQP| + |\angle PQR| + |\angle RQB| &= 180^\circ \\
|\angle PQR| &= 180^\circ - |\angle AQP| - |\angle RQB| \\
|\angle CAB| &= 180^\circ - 2|\angle AQP| \\
|\angle CBA| &= 180^\circ - 2|\angle RQB| \\
\Rightarrow |\angle CAB| + |\angle CBA| &= 360^\circ - 2[|\angle AQP| + |\angle RQB|] \\
\Rightarrow \frac{1}{2}[|\angle CAB| + |\angle CBA|] &= 180^\circ - |\angle AQP| - |\angle RQB| = |\angle PQR|
\end{aligned}$$

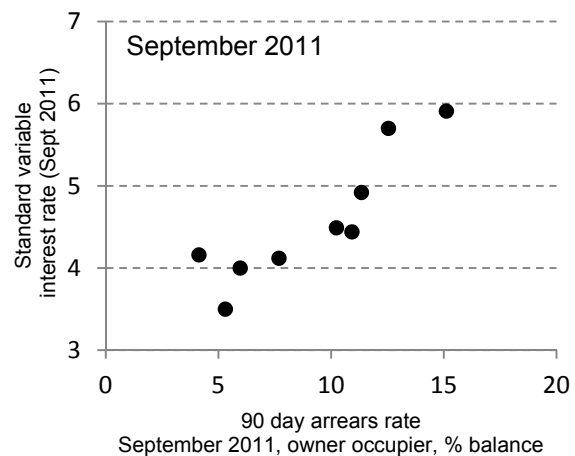
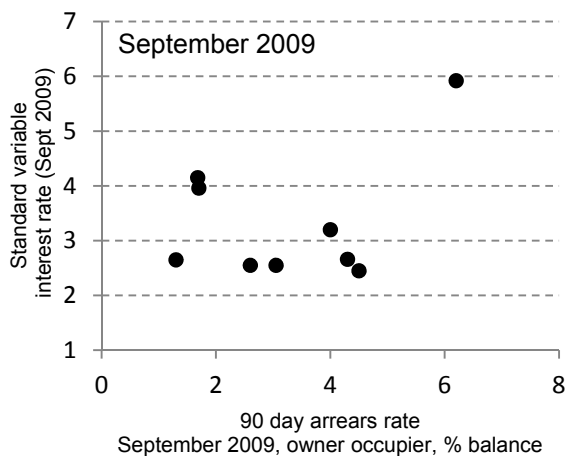
Answer Question 7 and Question 8.

Question 7

(75 marks)

To buy a home, people usually take out loans called *mortgages*. If one of the repayments is not made on time, the mortgage is said to be *in arrears*. One way of considering how much difficulty the borrowers in a country are having with their mortgages is to look at the percentage of all mortgages that are in arrears for 90 days or more. For the rest of this question, the term *in arrears* means in arrears for 90 days or more.

The two charts below are from a report about mortgages in Ireland. The charts are intended to illustrate the connection, if any, between the percentage of mortgages that are in arrears and the interest rates being charged for mortgages. Each dot on the charts represents a group of people paying a particular interest rate to a particular lender. The arrears rate is the percentage in arrears.



(Source: Goggin et al. *Variable Mortgage Rate Pricing in Ireland*, Central Bank of Ireland, 2012)

(a) Paying close attention to the scales on the charts, what can you say about the change from September 2009 to September 2011 with regard to:

(i) the arrears rates?

They've gone up a lot – they were mostly between 1 and 5 in 2009, and mostly between 5 and 15 in 2011.

(ii) the rates of interest being paid?

They've gone up a lot too – they were mostly between 2.3 and 4.1% in 2009, and mostly between 4 and 6% in 2011.

(iii) the relationship between the arrears rate and the interest rate?

There appears to be a stronger relationship 2011 than in 2009.

(b) What additional information would you need before you could estimate the median interest rate being paid by mortgage holders in September 2011?

You would need to know how many mortgage holders are represented by each point on the relevant diagram.

(c) Regarding the relationship between the arrears rate and the interest rate for September 2011, the authors of the report state: “The direction of causality ... is important” and they go on to discuss this.

Explain what is meant by the “direction of causality” in this context.

It is a question of whether higher arrears rates cause interest rates to go up, or whether higher interest rates cause arrears rates to go up, (assuming there is a causal relationship at all).

(d) A property is said to be in “negative equity” if the person owes more on the mortgage than the property is worth. A report about mortgaged properties in Ireland in December 2010 has the following information:

- Of the 475 136 properties examined, 145 414 of them were in negative equity.
- Of the ones in negative equity, 11 644 were in arrears.
- There were 317 355 properties that were neither in arrears nor in negative equity.

(i) What is the probability that a property selected at random (from all those examined) will be in negative equity?
Give your answer correct to two decimal places.

$$\frac{145414}{475136} = 0.30604711 = 0.31 \text{ (to two decimal places)}$$

(ii) What is the probability that a property selected at random from all those in negative equity will also be in arrears?
Give your answer correct to two decimal places.

$$\frac{11644}{145414} = 0.08007482 = 0.08 \text{ (to two decimal places)}$$

(iii) Find the probability that a property selected at random from all those in arrears will also be in negative equity.
Give your answer correct to two decimal places.

| | | | |
|-----------|--------------|---------------|---------------|
| | arrears | ¬arrears | total |
| neg. eq. | 11644 | 133770 | 145414 |
| ¬neg. eq. | 12367 | 317355 | 329722 |
| total | 24011 | 451125 | 475136 |

$\frac{11644}{24011} = 0.4849 = 0.48 \text{ (to two decimal places)}$

The Venn diagram consists of two overlapping ovals. The left oval is labeled 'arrears' and contains the number 12367. The right oval is labeled 'neg. eq.' and contains the number 133770. The overlapping region between the two ovals contains the number 11644. Below the right oval, the number 317355 is written, representing the total number of properties in both categories.

OR

$$P(A|N) = \frac{P(A \cap N)}{P(N)} \Rightarrow 0.08007 = \frac{P(A \cap N)}{0.30604} \Rightarrow P(A \cap N) = 0.0245$$

$$\text{But } P(A) = \frac{24011}{475136} = 0.05053$$

$$P(N|A) = \frac{P(N \cap A)}{P(A)} = \frac{0.0245}{0.05053} = 0.4848 = 0.48 \text{ (to two decimal places)}$$

- (e) The study described in part (d) was so large that it can be assumed to represent the population. Suppose that, in early 2012, researchers want to know whether the proportion of properties in negative equity has changed. They analyse 2000 randomly selected properties with mortgages. They discover that 552 of them are in negative equity. Use a hypothesis test at the 5% level of significance to decide whether there is sufficient evidence to conclude that the situation has changed since December 2010.

Be sure to state the null hypothesis clearly, and to state the conclusion clearly.

Null hypothesis: proportion in negative equity unchanged: $p = 0.31$.

Alternative hypothesis: it has changed: $p \neq 0.31$.

95% margin of error for samples of size 2000 is $\frac{1}{\sqrt{2000}} \approx 0.0224$

So, reject null hypothesis if observed proportion lies outside 0.31 ± 0.0224 .

Observed proportion = $\frac{552}{2000} \approx 0.276$.

$0.276 \notin [0.2876, 0.3224]$

Outside margin of error, so reject null hypothesis.

The proportion in negative equity has changed.

OR

Null hypothesis: proportion in negative equity unchanged: $p = 0.31$.

95% margin of error for samples of size 2000 is $\frac{1}{\sqrt{2000}} \approx 0.0224$

Observed proportion = $\frac{552}{2000} \approx 0.276$.

\therefore the 95% confidence interval for the population proportion is:

$0.276 - 0.0224 < p < 0.276 + 0.0224$

$0.2536 < p < 0.2984$

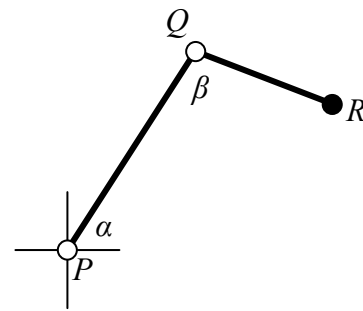
0.31 outside this range.

Therefore reject null hypothesis. Proportion in negative equity has changed.

Question 8

(75 marks)

The diagram is a representation of a robotic arm that can move in a vertical plane. The point P is fixed, and so are the lengths of the two segments of the arm. The controller can vary the angles α and β from 0° to 180° .



- (a) Given that $|PQ| = 20$ cm and $|QR| = 12$ cm, determine the values of the angles α and β so as to locate R , the tip of the arm, at a point that is 24 cm to the right of P , and 7 cm higher than P . Give your answers correct to the nearest degree.

$|PR|^2 = 7^2 + 24^2$
 $|PR| = 25$

$25^2 = 20^2 + 12^2 - 2(20)(12)\cos\beta$
 $\cos\beta = -0.16875$
 $\beta \approx 100^\circ$

$12^2 = 25^2 + 20^2 - 2(25)(20)\cos(\alpha - \gamma)$
 $\cos(\alpha - \gamma) = 0.881$
 $\alpha - \gamma \approx 28.237^\circ$

$\tan\gamma = \frac{7}{24}$
 $\gamma \approx 16.260^\circ$

$\therefore \alpha \approx 44^\circ$

- (b) In setting the arm to the position described in part (a), which will cause the greater error in the location of R : an error of 1° in the value of α or an error of 1° in the value of β ?

Justify your answer. You may assume that if a point moves along a circle through a small angle, then its distance from its starting point is equal to the length of the arc travelled.

Ans: α

Reason: 1° error in α causes R to move along an arc of radius 25.
 1° error in β causes R to move along an arc of radius 12.

So, since $l = r\theta$, and θ is the same in each case, the point moves further in the first case.

- (c) The answer to part (b) above depends on the particular position of the arm. That is, in certain positions, the location of R is more sensitive to small errors in α than to small errors in β , while in other positions, the reverse is true. Describe, with justification, the conditions under which each of these two situations arises.

More sensitive to errors in α when $|PR| > 12$

More sensitive to errors in β when $|PR| < 12$

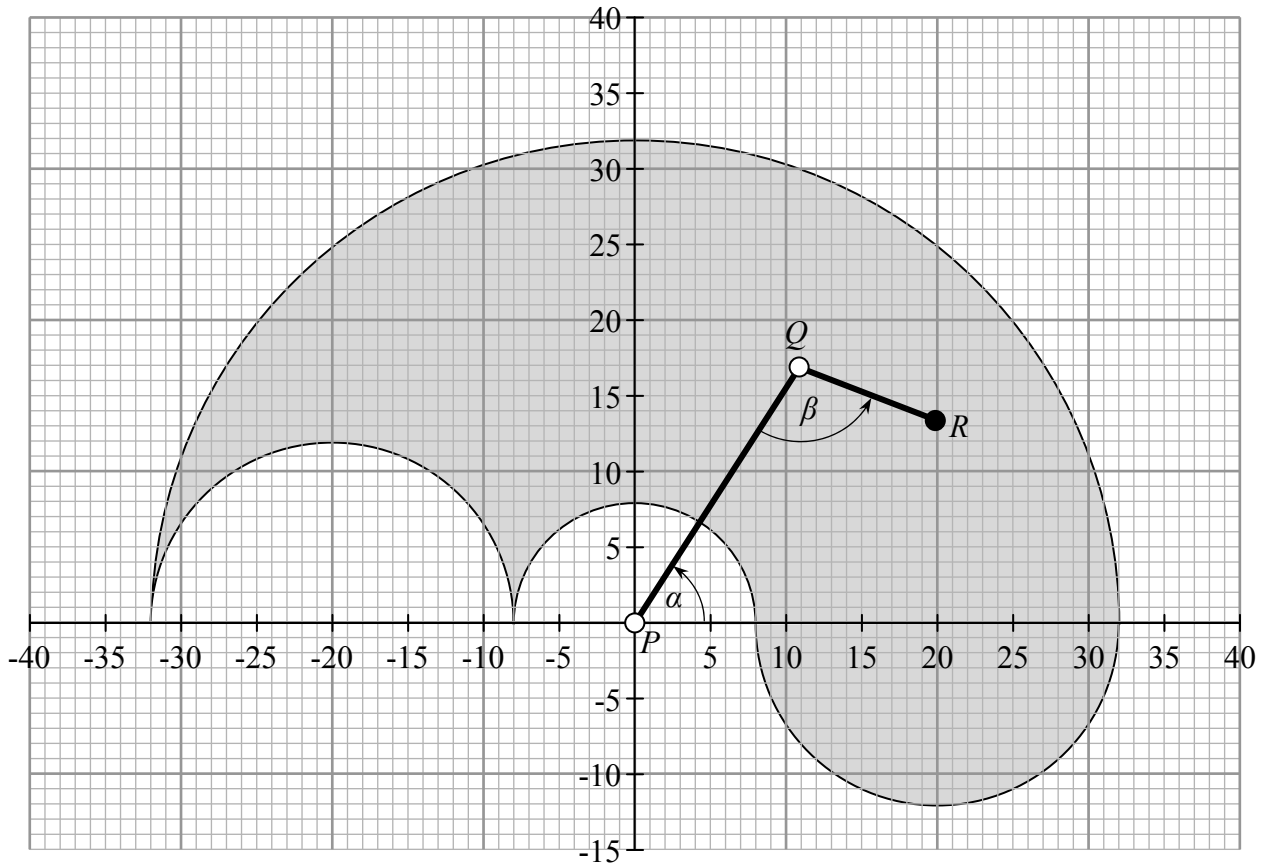
The condition $|PR| > 12$

is true whenever

$$\beta > \cos^{-1}\left(\frac{5}{6}\right) \approx 33.6^\circ$$

(Borderline case is when ΔPQR is isosceles with $|QR| = |RP|$.)

- (d) Illustrate the set of all possible locations of the point R on the coordinate diagram below. Take P as the origin and take each unit in the diagram to represent a centimetre in reality. Note that α and β can vary only from 0° to 180° .



Marking Scheme – Paper 2

Structure of the marking scheme

Candidate responses are marked according to different scales, depending on the types of response anticipated. Scales labelled A divide candidate responses into two categories (correct and incorrect). Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on. The scales and the marks that they generate are summarised in this table:

| Scale label | A | B | C | D | E |
|------------------|---|----------|--------------|-------------------|----------------------|
| No of categories | 2 | 3 | 4 | 5 | 6 |
| 5 mark scale | | 0, 3, 5 | 0, 3, 4, 5 | | |
| 10 mark scale | | 0, 5, 10 | 0, 4, 8, 10 | | |
| 15 mark scale | | | | | |
| 20 mark scale | | | 0, 7, 18, 20 | 0, 7, 10, 18, 20 | |
| 25 mark scale | | | | 0, 15, 20, 22, 25 | 0, 5, 10, 15, 20, 25 |

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)

- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)

- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

E-scales (six categories)

- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response almost half-right (lower middle partial credit)
- response more than half-right (upper middle partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving incorrect rounding or omission of units, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk. Thus, for example, *scale 10C** indicates that 9 marks may be awarded.

Summary of mark allocations and scales to be applied

Section A

Question 1

- (a) 5B, 5B, 5B
- (b) 10C

Question 2

- (a) 5B, 5B
- (b) 5C
- (c) 5B
- (d) 5B

Question 3

25D

Question 4

- (a) 5C
- (b)(i) 10C*
- (b)(ii) 10C*

Question 5

- (a) 20D
- (b) 5C

Question 6A

- (a)(i) 10C
- (a)(ii) 5C
- (b) 10C

Question 6B

25E

Section B

Question 7

- (a)(i) 5B
- (a)(ii) 5B
- (a)(iii) 10B
- (b) 5B
- (c) 5C
- (d)(i) 10C*
- (d)(ii) 5B*
- (d)(iii) 10C*
- (e) 20D

Question 8

- (a)| PR | 10B
- (a) β 20C*
- (a) α 25D*
- (b) 5C
- (c) 5C
- (d) 10C

Detailed marking notes

Section A

Question 1

(a) Scale 5B, 5B, 5B (0, 3, 5)

Partial Credit:

- Incomplete statement of method (with some merit)

(b) Scale 10C (0, 4, 8, 10).

Low Partial Credit:

- Any reasonable first step.

High Partial Credit:

- Correct method applied with some error(s)
- Correct method but vertices not taken in correct order (i.e *ABDC* or equivalent)
- Correct method completed but with no conclusion.

Question 2

(a) Scale 5B, 5B (0, 3, 5)

Partial Credit:

- Centre or radius found
- Incomplete statement of method (with some merit) e.g., completing square

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Formula for distance between two centres with some substitution
- Difference between radii found or implied

High Partial Credit:

- No conclusion or incorrect conclusion

(c) Scale 5B (0, 3, 5)

Partial Credit:

- (4, 7) substituted into one circle
- Subtracts both equations and stops
- (4, 7) substituted into both circles with an error
- Finds common tangent and shows that (4, 7) is on it, without showing that it is also on one of the circles

(d) Scale 5B (0, 3, 5)

Partial Credit:

- Some relevant slope found
- (4, 7) inserted into equation of line formula but without slope found
- Equation of line but (4, 7) incorrectly inserted

Question 3

Scale 25D (0, 15, 20, 22, 25)

Low Partial Credit:

- Any reasonable first step, such as:
 - radius / diameter indicated to one of the points of contact
 - intercepts of either tangent on axes indicated
 - $(1, 1)$ and/or (k, k) on diagram with no further work of merit
 - $|g| = |f|$ or $g = \pm f$
 - centre $(-g, -g)$ or equivalent
- Perpendicular distance of centre to either tangent indicated

Mid Partial Credit:

- Centre (r, r)
- Equation connecting r and k (i.e. work towards $2r - 1 = k$)
- Writes equation of circle $(x - r)^2 + (y - r)^2 = r^2$
- Equation $x^2 + y^2 + 2gx + 2gy + g^2 = 0$ or equivalent
- $y = x$ and further work of merit
- States $g = f$ and $g^2 = f^2 = c$
- Perpendicular distance $(-g, -g)$ to tangent
- Substantive work at finding equation of a relevant angle-bisector, other than $y = x$

High Partial Credit:

- $r = 2 + \sqrt{2}$ or equivalent but fails to finish

Question 4

(a) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- One or both 'given' assumptions stated or implied

High Partial Credit:

- Either 'independence' or 'probability of success the same each time' stated.

(b)(i) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit:

- Any first step e.g. reference to 0.4 or equivalent

High Partial Credit:

- Correct expression
- Answer with one error in components

Note: Rounding incomplete: 9 marks

(b)(ii) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit:

- Reference to 0.6 or equivalent for fifth shot

High Partial Credit:

- Correct expression
- Answer with one error in components

Note: Rounding incomplete: 9 marks

Question 5

(a) Scale 20D (0, 7, 10, 18, 20)

Low Partial Credit:

- Any relevant step
- Some relevant diagram

Mid Partial Credit:

- Reference to 2.5
- $P(Z > 2.5) = 0.0062$ and stops

High Partial Credit:

- $P(|Z| > 2.5) = 0.0124$
- Correct method with some error

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Any relevant step
- Some relevant diagram
- One case taken only

High Partial Credit:

- Probability of both situations calculated but fails to complete fully

Question 6A

(a)(i) Scale 10C (0, 4, 8, 10)

Low Partial Credit:

- Any correct step

High Partial Credit:

- Correct method but outside the tolerance of 2°

(a)(ii) Scale 5C (0, 3, 4, 5)

Low Partial Credit:

- Any correct step

High Partial Credit:

- Correct method but outside the tolerance of 2°

(b) Scale 10C (0, 4, 8, 10)

Low Partial Credit:

- Any correct step e.g.,
 - identifies two equal sides
 - identifies two equal angles
 - extends DE to intersect l_1

High Partial Credit:

- Proof with correct steps but without justification of steps
- One error in establishing congruence

Question 6B

Scale 25E (0, 5, 10, 15, 20, 25)

Low Partial Credit:

- Any correct statement

Lower Middle Partial Credit:

- Some substantive work towards proof e.g. at least one full step complete
- Two distinct relevant statements

Upper Middle Partial Credit:

- Substantive proof with two critical steps missing

High Partial Credit:

- Correct proof but critical step missing
- Correct proof without justification of steps

Question 7

(a)(i) Scale 5B (0, 3, 5)

Partial Credit

- Incomplete statement e.g., “it has changed”.

(a)(ii) Scale 5B (0, 3, 5)

Partial Credit

- Incomplete or partly correct statement

(a)(iii) Scale 10B (0, 5, 10)

Partial Credit

- Incomplete or partly correct statement, e.g.,
 - “They have changed”
 - “Closer to being a line”
 - Reference to positive

(b) Scale 5B (0, 3, 5)

Partial Credit

- Some reference to the middle household

Note: accept (for full credit) reference to needing information about mortgage holders who are not on standard variable rates.

(c) Scale 5C (0, 3, 4, 5)

Low Partial Credit

- Both may be caused by something else
- General statement regarding correlation not implying causality – no context

High Partial Credit

- No reference to reverse situation, (e.g.: “It’s about whether high interest rates cause high arrears rates or not.”)
- Correct interpretation of concept, but not contextualised, (e.g. “It’s a question of which variable causes which.”)

(d)(i) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit

- Uses a relevant number
- Writes $\frac{\#E}{\#S}$ or equivalent.
- Identifies “number of outcomes of interest = ...” or “total number of outcomes = ...”.

High Partial Credit

- Answer in the form of a fraction

(d)(ii) Scale 5B* (0, 3, [4], 5)

Partial Credit

- Uses a relevant number
- Writes $\frac{\#E}{\#S}$ or equivalent.
- Identifies “number of outcomes of interest = ...” or “total number of outcomes = ...”.
- Answer in form of a fraction

(d)(iii) Scale 10C* (0, 4, 8, [9], 10)

Low Partial Credit

- Attempt to combine (i) and (ii) for part (iii)
- Calculates total arrears and stops

High Partial Credit

- Answer as a fraction
- $\frac{11644}{12367} = 0.9415 = 0.94$

(e) Scale 20D (0, 7, 10, 18, 20)

Low Partial Credit

- One relevant step e.g. null hypothesis stated only
- Margin of error or observed proportion and does not continue

Mid Partial Credit:

- Substantive work with one or more critical omissions
- Margin of error and observed proportion found but fails to continue

High Partial Credit

- Failure to state null hypothesis correctly and/or failure to contextualise answer (e.g., stops at “Reject the null hypothesis”).

Question 8

(a) $|PR|$ Scale 10B (0, 5, 10)

Partial Credit

- Some use of Pythagoras

(a) β Scale 20C* (0, 7, 18, [19], 20)

Low Partial Credit

- Cosine Rule with some substitution

High Partial Credit

- $\cos \beta$ calculated

(a) α Scale 25D* (0, 15, 20, 22, [24], 25)

Low Partial Credit

- Some work towards solving required angle with *sine* or *cosine* rule
- $\tan \gamma = \frac{7}{24}$ with no work towards $\alpha - \gamma$

Middle Partial Credit

- $\cos \alpha - \gamma$ found

High Partial Credit

- $\alpha - \gamma$ and γ calculated but α not evaluated

(b) Scale 5C (0, 3, 4, 5)

Low Partial Credit

- Effort at working out values for angles
- Correct answer without justification

High Partial Credit

- Correct answer without being fully justified

(c) Scale 5C (0, 3, 4, 5)

Low Partial Credit

- Some reference to distance between P and R
- Treats as percentage error in angles, rather than absolute error in location. e.g. "If α is smaller than β , then a 1° error in α is a bigger percentage error than a 1° error in β ."

High Partial Credit

- One situation only dealt with correctly
- Clearly understands concept that the radius of the rotation is the determining factor, but makes error(s) in explanation (e.g. mixes up distances involved).

(d) Scale 10C (0, 4, 8, 10)

Low Partial Credit

- Any relevant semi circle sketched or implied

High Partial Credit

- Any correct semi circle inserted in addition to semicircle centre P with radius 32.

Marcanna Breise as ucht Freagairt trí Ghaeilge

(Bonus marks for answering through Irish)

Ba chóir marcanna de réir an ghnáthráta a bhronnadh ar iarrthóirí nach ngnóthaíonn níos mó ná 75% d'iomlán na marcanna don pháipéar. Ba chóir freisin an marc bónais sin a shlánú **síos**.

Déantar an cinneadh agus an ríomhaireacht faoin marc bónais i gcás gach páipéir ar leithligh.

Is é 5% an gnáthráta agus is é 300 iomlán na marcanna don pháipéar. Mar sin, bain úsáid as an ngnáthráta 5% i gcás iarrthóirí a ghnóthaíonn 225 marc nó níos lú, e.g. $198 \text{ marc} \times 5\% = 9.9 \Rightarrow$ bónas = 9 marc.

Má ghnóthaíonn an t-iarrthóir níos mó ná 225 marc, ríomhtar an bónas de réir na foirmle $[300 - \text{bunmharc}] \times 15\%$, agus an marc bónais sin a shlánú **síos**. In ionad an ríomhaireacht sin a dhéanamh, is féidir úsáid a bhaint as an tábla thíos.

| Bunmharc | Marc Bónais |
|-----------|-------------|
| 226 | 11 |
| 227 – 233 | 10 |
| 234 – 240 | 9 |
| 241 – 246 | 8 |
| 247 – 253 | 7 |
| 254 – 260 | 6 |
| 261 – 266 | 5 |
| 267 – 273 | 4 |
| 274 – 280 | 3 |
| 281 – 286 | 2 |
| 287 – 293 | 1 |
| 294 – 300 | 0 |

